Since the dot product of vectors is distributive, therefore, if $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$, then

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j).$$

In particular, if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors perpendicular to one another, then $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ so that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

which is the familiar expression for the evaluation of the dot product in terms of the vector components.

(c) *Factoring*: If

$$T_{ij} n_j - \lambda n_i = 0,$$

then, using the Kronecker delta, we can write $n_i = \delta_{ij} n_j$, so that we have

$$T_{ij} n_j - \lambda \delta_{ij} n_j = 0.$$

Thus,

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0.$$

(d) *Contraction*: The operation of identifying two indices is known as a *contraction*. Contraction indicates a sum on the index. For example, T_{ii} is the contraction of T_{ij} with

$$T_{ii} = T_{11} + T_{22} + T_{33}.$$

If

$$T_{ij} = \lambda \Delta \delta_{ij} + 2\mu E_{ij},$$

then

$$T_{ii} = \lambda \Delta \delta_{ii} + 2\mu E_{ii} = 3\lambda \Delta + 2\mu E_{ii}.$$

PROBLEMS FOR PART A

2.1 Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

evaluate (a) S_{ii}, (b) $S_{ij} S_{ij}$, (c) $S_{ji} S_{ji}$, (d) $S_{jk} S_{kj}$, (e) $a_m a_m$, (f) $S_{mn} a_m a_n$, and (g) $S_{nm} a_m a_n$.

2.2 Determine which of these equations has an identical meaning with $a_i = Q_{ij} a'_j$.
(a) $a_p = Q_{pm} a'_m$, (b) $a_p = Q_{qp} a'_q$, (c) $a_m = a'_n Q_{mn}$.

2.3 Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix},$$

demonstrate the equivalence of the subscripted equations and the corresponding matrix equations in the following two problems:

(a) $b_i = B_{ij} a_j$ and $[b] = [B][a]$ and (b) $s = B_{ij} a_i a_j$ and $s = [a]^{\mathrm{T}} [B][a]$.

2.4 Write in indicial notation the matrix equation (a) $[A]=[B][C]$, (b) $[D]=[B]^{T}[C]$ and (c) $[E]=[B]^{T}[C][F]$.

2.5 Write in indicial notation the equation (a) $s = A_1^2 + A_2^2 + A_3^2$ and (b) $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0$.

2.6 Given that $S_{ij}=a_i a_j$ and $S'_{ij} = a'_i a'_j$, where $a'_i = Q_{mi} a_m$ and $a'_j = Q_{nj} a_n$, and $Q_{ik} Q_{jk} = \delta_{ij}$, show that $S'_{ii} = S_{ii}$.

2.7 Write $a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$ in long form.

2.8 Given that $T_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}$, show that
(a) $T_{ij} E_{ij} = 2\mu E_{ij} E_{ij} + \lambda (E_{kk})^2$ and (b) $T_{ij} T_{ij} = 4\mu^2 E_{ij} E_{ij} + (E_{kk})^2 (4\mu\lambda + 3\lambda^2)$.

2.9 Given that $a_i = T_{ij} b_j$, and $a'_i = T'_{ij} b'_j$, where $a_i = Q_{im} a'_m$ and $T_{ij} = Q_{im} Q_{jn} T'_{mn}$,
(a) show that $Q_{im} T'_{mn} b'_n = Q_{im} Q_{jn} T'_{mn} b_j$ and (b) if $Q_{ik} Q_{im} = \delta_{km}$, then $T'_{kn}(b'_n - Q_{jn} b_j) = 0$.

2.10 Given

$$[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},$$

evaluate $[d_i]$, if $d_k = \varepsilon_{ijk} a_i b_j$, and show that this result is the same as $d_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_k$.

2.11 (a) If $\varepsilon_{ijk} T_{ij} = 0$, show that $T_{ij} = T_{ji}$, and (b) show that $\delta_{ij} \varepsilon_{ijk} = 0$.

2.12 Verify the following equation: $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$. *Hint:* There are six cases to be considered: (1) $i = j$, (2) $i = k$, (3) $i = l$, (4) $j = k$, (5) $j = l$, and (6) $k = l$.

2.13 Use the identity $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ as a shortcut to obtain the following results: (a) $\varepsilon_{ilm} \varepsilon_{jlm} = 2\delta_{ij}$ and (b) $\varepsilon_{ijk} \varepsilon_{ijk} = 6$.

2.14 Use the identity $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ to show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

2.15 Show that (a) if $T_{ij} = -T_{ji}$, then $T_{ij} a_i a_j = 0$, (b) if $T_{ij} = -T_{ji}$, and $S_{ij} = S_{ji}$, then $T_{ij} S_{ij} = 0$.

2.16 Let $T_{ij} = \frac{1}{2}(S_{ij} + S_{ji})$ and $R_{ij} = \frac{1}{2}(S_{ij} - S_{ji})$, show that $T_{ij} = T_{ji}, R_{ij} = -R_{ji}$, and $S_{ij} = T_{ij} + R_{ij}$.

2.17 Let $f(x_1, x_2, x_3)$ be a function of x_1, x_2, and x_3 and let $v_i(x_1, x_2, x_3)$ be three functions of x_1, x_2, and x_3. Express the total differential df and dv_i in indicial notation.

2.18 Let $|A_{ij}|$ denote the determinant of the matrix $[A_{ij}]$. Show that $|A_{ij}| = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$.

PART B: TENSORS

2.6 TENSOR: A LINEAR TRANSFORMATION

Let $\mathbf{T}$ be a transformation that transforms any vector into another vector. If $\mathbf{T}$ transforms $\mathbf{a}$ into $\mathbf{c}$ and $\mathbf{b}$ into $\mathbf{d}$, we write $\mathbf{Ta} = \mathbf{c}$ and $\mathbf{Tb} = \mathbf{d}$.

If $\mathbf{T}$ has the following linear properties:

$$\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb}, \tag{2.6.1}$$

$$\mathbf{T}(\alpha \mathbf{a}) = \alpha \mathbf{Ta}, \tag{2.6.2}$$